



The Reconstruction Problem Revisited

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THE RECONSTRUCTION PROBLEM REVISITED

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Abstract

The role of reconstruction in avoiding oscillations in upwind schemes is reexamined, with the aim of providing simple, concise proofs. In one dimension, it is shown that if the reconstruction is any arbitrary function bounded by neighboring cell averages and increasing within a cell for increasing data, the resulting scheme is monotonicity preserving, even though the reconstructed function may have overshoots and undershoots at the cell edges and is in general not a monotone function. In the special case of linear reconstruction, it is shown that merely bounding the reconstruction between neighboring cell averages is sufficient to obtain a monotonicity preserving scheme.

In two dimensions, it is shown that some 1D TVD limiters applied in each direction result in schemes that are not positivity preserving, i.e. do not give positive updates when the data are positive. A simple proof is given to show that if the reconstruction inside the cell is bounded by the neighboring cell averages (including corner neighbors), then the scheme is positivity preserving. A new limiter that enforces this condition but is not as dissipative as the Minmod limiter is also presented.

Introduction

In this paper, we begin by reexamining the reconstruction step in upwind schemes¹⁻⁶ with the aim of deriving slightly more general results and providing simple, concise proofs.

To introduce the one dimensional problem, consider the model equation

$$u_t + u_x = 0, \quad (1)$$

where t is time, x is distance, and $u(x, 0) = u_0(x)$ is the initial condition. We denote by \bar{u}_j^n the cell averages of $u(x, t^n)$ on a uniform grid of width Δx and cell center $x_j = j\Delta x$,

$$\bar{u}_j^n = \frac{1}{\Delta x} \int_{x_j - \Delta x/2}^{x_j + \Delta x/2} u(x, t^n) dx. \quad (2)$$

Let $R_j^n(\xi)$, $\xi = (x - x_j)/\Delta h$ denote the reconstruction of $u(x, t^n)$ in the cell j expressed in local coordinates. The new cell averages are obtained by convecting the reconstructed profiles a time step and averaging the resulting profiles over a cell. The result is

$$\bar{u}_j^{n+1} = \int_{1/2-\lambda}^{1/2} R_{j-1}^n(\xi) d\xi + \int_{-1/2}^{1/2-\lambda} R_j^n(\xi) d\xi \quad (3)$$

where $\lambda = \Delta t/h$. For clarity, we omit the superscript n when there is no confusion, i.e. \bar{u}_j denotes \bar{u}_j^n .

For increasing data, i.e. $\bar{u}_j \geq \bar{u}_{j-1}$ for all j , we are interested in schemes for which the new cell averages are also increasing. Such schemes are called monotonicity-preserving schemes. They mimic the exact solution and as a special case it follows that a step function propagates without spurious oscillations.

The question we are interested in is:

What are sufficient conditions on the reconstruction $R_j(\xi)$ that will ensure that the scheme (3) is monotonicity-preserving ?

This question has been studied extensively over the last two decades in the upwind literature, and we review existing conditions below. Our aim is to provide a new, simple, concise proof that covers the general case. Generality is achieved here by considering a) completely arbitrary functions for reconstruction and b) by considering Monotonicity-Preserving (MP) schemes as opposed to Total Variation Diminishing (TVD) schemes. TVD schemes are monotonicity-preserving, but not vice-versa. In addition, MP schemes can be designed to avoid the chronic loss of accuracy at smooth extrema incurred by TVD schemes (see Ref. 7 for an example).

Existing Conditions

There are two different conditions stated in the literature. The first condition, introduced by Van Leer¹ in his seminal paper on upwind schemes and often called Van Leer's condition,

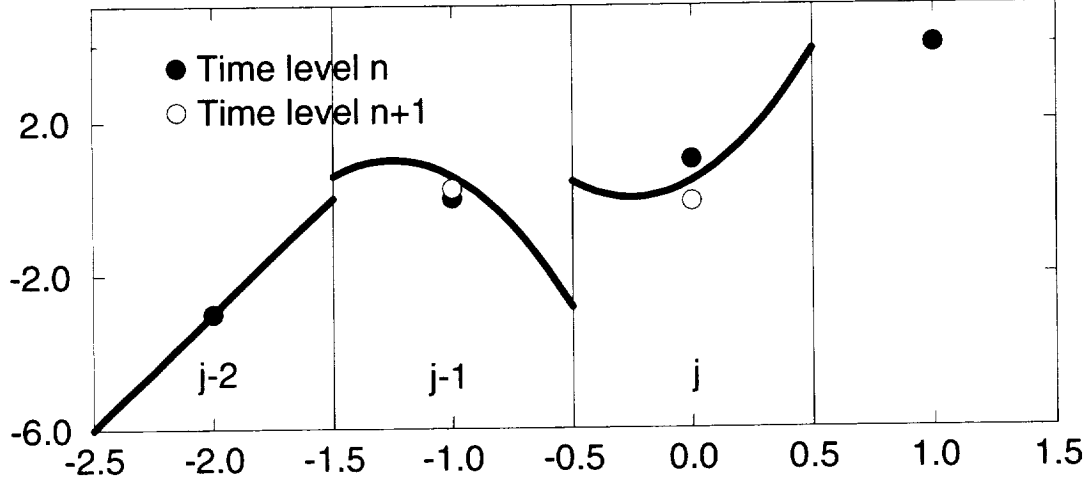


Figure 1: An oscillation created by reconstructions bounded by neighboring cell averages. $\lambda = 0.25$.

states that the reconstruction $R_j(\xi)$ must be bounded by the neighboring cell averages, i.e. for increasing data, the condition is

$$\bar{u}_{j-1} \leq R_j(\xi) \leq \bar{u}_{j+1} \quad (4)$$

While this condition suffices for linear reconstructions, we will show by an explicit counterexample that this is not sufficient for general reconstructions. Indeed, let $\bar{u}_{j-3} = -6$, $\bar{u}_{j-2} = -3$, $\bar{u}_{j-1} = 0$, $\bar{u}_j = 1$, $\bar{u}_{j+1} = 4$, and the reconstructions be $R_{j-2}(\xi) = -3 + 6\xi$; $R_{j-1}(\xi) = (4 - 24\xi - 48\xi^2)/7$; $R_j(\xi) = 1 - (4 - 24\xi - 48\xi^2)/7$; $R_{j+1}(\xi) = 4$. These reconstructions are plotted in Fig. 1. Note that the data is increasing and (4) is satisfied in each cell. However, the cell averages at the next time step given by (3) for $\lambda = 1/4$ are $\bar{u}_{j-1}^{n+1} = 27/112$, $\bar{u}_j^{n+1} = -3/28$, and $\bar{u}_{j+1}^{n+1} = 103/28$. So monotonicity is not preserved.

Another condition (Harten et. al², LeVeque³) is stated in terms of the total variation of the reconstruction. Let $TV(f(x))$ denote the total variation of $f(x)$ given by

$$TV(f) = \int_{-\infty}^{+\infty} |f'(x)| dx \quad (5)$$

and $TV(\bar{u})$ denote the discrete total variation of the cell averages given by

$$TV(\bar{u}) = \sum_{k=-\infty}^{+\infty} |\bar{u}_{k+1} - \bar{u}_k| \quad (6)$$

Then the required condition on the reconstruction is that

$$TV(R(.)) \leq TV(\bar{u}). \quad (7)$$

This condition holds for any general reconstruction. For increasing data, (7) implies that $R(\cdot)$ is an increasing function with no overshoots at the cell faces. However, designing schemes that satisfy this condition is a bit more difficult because the conditions on $R_j(\xi)$ now involve $R_{j-1}(\xi)$ and $R_{j+1}(\xi)$ in addition to the cell averages. This condition is also unnecessarily restrictive since most of the well known TVD schemes do not satisfy it.

For example, consider the data given by $\bar{u}_{j-1} = 0$, $\bar{u}_j = 0.2$, $\bar{u}_{j+1} = 0.8$, and $\bar{u}_{j+2} = 2.6$. The reconstructions obtained by the popular Superbee limiter and Van Leer's Average limiter ((45) and (46) in the appendix) are identical and given by $R_j(\xi) = 0.2 + 0.4\xi$ and $R_{j+1}(\xi) = 0.8 + 1.2\xi$, which has an overshoot at the face $x_{j+1/2}$. Thus both these schemes violate (7) although both limiters are monotonicity preserving. This example clearly illustrates that condition (7) is unnecessarily restrictive for monotonicity preservation.

Thus, of the two conditions available in the literature, neither are completely satisfactory. Van Leer's condition fails for general reconstructions while the second condition (7) is so restrictive that many well known schemes do not satisfy it. In the next section, we introduce a slight modification of Van Leer's condition that is more general than (7) and for which monotonicity preservation can be proved for general $R_j(\xi)$.

A Modified Condition

First, we need two inequalities that hold for increasing functions. A function $f(x)$ is an *increasing function* if for any $x_1 > x_2$, $f(x_1) \geq f(x_2)$. Let $f(x)$ be an increasing function in $[-1/2, 1/2]$ and let its mean over this interval be denoted by \bar{f} . Then, for $0 \leq \lambda \leq 1$, we have the following two inequalities that are useful.

$$\frac{1}{\lambda} \int_{1/2-\lambda}^{1/2} f(\xi) d\xi \geq \bar{f}, \quad \frac{1}{1-\lambda} \int_{-1/2}^{1/2-\lambda} f(\xi) d\xi \leq \bar{f}. \quad (8)$$

The first inequality states that sampling an increasing function from the right yields an average greater than the cell average. To prove this inequality, note that if we denote by $g(\lambda)$ the left hand side of the first inequality, we get $\lambda g'(\lambda) = f(1/2 - \lambda) - g(\lambda)$. As this is nonpositive for all λ , the inequality follows. The proof of the second inequality is similar.

The modified condition can be stated as follows:

Theorem-1: Assume we have increasing data and the reconstructions $R_j(\xi)$ satisfy the three conditions given below for all j :

$$\overline{R_j(\xi)} = \bar{u}_j \quad (9)$$

$$\bar{u}_{j-1} \leq R_j(\xi) \leq \bar{u}_{j+1} \quad (10)$$

$$R_j(\xi) \text{ is an increasing function} \quad (11)$$

Then, for $0 \leq \lambda \leq 1$, the cell averages at the next time step are also increasing and (3) becomes a monotonicity preserving scheme.

Proof:

Since $R_{j-1}(\xi)$ is an increasing function, the first integral in (3) satisfies

$$\frac{1}{\lambda} \int_{1/2-\lambda}^{1/2} R_{j-1}(\xi) d\xi \geq \bar{u}_{j-1} \quad (12)$$

and from (10) the second integral satisfies

$$\int_{-1/2}^{1/2-\lambda} R_j(\xi) d\xi \geq (1-\lambda)\bar{u}_{j-1} \quad (13)$$

Thus $\bar{u}_j^{n+1} \geq \bar{u}_{j-1}$. Similarly, from the bounds on $R_{j-2}(\xi)$, we have

$$\int_{1/2-\lambda}^{1/2} R_{j-2}(\xi) d\xi \leq \lambda \bar{u}_{j-1} \quad (14)$$

and since $R_{j-1}(\xi)$ is an increasing function,

$$\frac{1}{1-\lambda} \int_{-1/2}^{1/2-\lambda} R_{j-1}(\xi) d\xi \leq \bar{u}_{j-1}. \quad (15)$$

So $\bar{u}_{j-1}^{n+1} \leq \bar{u}_{j-1}$. Hence, $\bar{u}_j^{n+1} \geq \bar{u}_{j-1}^{n+1}$ completing the proof. \square

Thus, if we add to Van Leer's condition the requirement that the reconstruction be increasing when the data are increasing, we can prove monotonicity preservation for any general reconstruction. Note that conditions (9), (10) and (11) are more general than (7) since $TV(R(.))$ can be greater than $TV(\bar{u})$. In addition, all the well known second order TVD schemes satisfy (9), (10) and (11) so that Theorem -1 is not unusually restrictive.

Although adding the requirement that the reconstruction be a monotone function is a fairly trivial modification, we have not seen Theorem - 1 stated and proved anywhere in the literature. Indeed, this is all the more surprising since the fairly restrictive condition (7) is widely quoted. Monotonicity of the reconstruction was imposed as a condition by Colella and Woodward⁸ in the design of their piecewise parabolic method (PPM), but monotonicity preservation is not proved there.

TVD Schemes

TVD schemes have some theoretical advantages over monotonicity preserving schemes such as a guaranteed convergence to a weak solution of the underlying conservation law². Monotonicity-preserving schemes differ from TVD schemes only in their treatment of

local extrema. It is not surprising, therefore, that with some recipe for reconstruction at an extrema, a result similar to Theorem - 1 can be proved for TVD schemes.

Theorem-2: *Let the reconstructions $R_j(\xi)$ satisfy the conditions given below for all j :*

$$\overline{R_j(\xi)} = \bar{u}_j; \quad (16)$$

$$\text{when } \bar{u}_{j-1} \leq \bar{u}_j \leq \bar{u}_{j+1} :$$

$$u_{j-1} \leq R_j(\xi) \leq \bar{u}_{j+1} \text{ and } R_j(\xi) \text{ is an increasing function}; \quad (17)$$

$$\text{when } \bar{u}_{j-1} \geq \bar{u}_j \geq \bar{u}_{j+1} :$$

$$\bar{u}_{j-1} \geq R_j(\xi) \geq \bar{u}_{j+1} \text{ and } R_j(\xi) \text{ is a decreasing function}; \quad (18)$$

$$\text{otherwise :}$$

$$R_j(\xi) = \bar{u}_j. \quad (19)$$

Then, for $0 \leq \lambda \leq 1$, (3) becomes a TVD scheme.

Proof: We introduce some terminology on data types. Let us say the data is increasing at j when $u_{j-1} \leq u_j \leq \bar{u}_{j+1}$ and decreasing at j when $\bar{u}_{j-1} \geq \bar{u}_j \geq \bar{u}_{j+1}$. From Theorem-1 it follows that for data increasing at j and $j - 1$, or decreasing at j and $j - 1$ the new cell averages lie inside the range defined by neighboring old cell averages, i.e.

$$\text{Min}(\bar{u}_j, \bar{u}_{j-1}) \leq \bar{u}_j^{n+1} \leq \text{Max}(\bar{u}_j, \bar{u}_{j-1}) \quad (20)$$

Consider the case where data is increasing at j but we have a local minimum at $j - 1$, i.e. $u_{j-1} \leq \bar{u}_{j-2}$ and $\bar{u}_{j-1} \leq \bar{u}_j$. In this case using (19) and (3) we can show that the bound (20) is still satisfied. Similarly, we can show that the bound (20) holds also for all other cases such as data decreasing at $j - 1$ and a minimum at j , or increasing at $j - 1$ and a maximum at j etc. In short, (20) holds for all data.

Since (20) holds for all data, no new extrema can be created and the value of a local maximum can only decrease and the value of a local minimum can only increase. Hence the total variation can only decrease and the theorem is proved. \square

The Linear Case

The linear case has been exhaustively studied ¹⁻⁶ in the context of TVD flux limiters and will not be repeated here. However, we wish to highlight the following peculiarity of the linear case, namely that monotonicity is preserved even without requiring that the reconstruction be increasing for increasing data, i.e. even without (11). In other words, for increasing data, the slopes can be decreasing and monotonicity is still preserved so long as

the linear reconstructions remain bounded by neighboring cell averages. For completeness, a proof is sketched out below.

Assume increasing data and linear reconstructions $R_j(\xi) = \bar{u}_j + s_j\xi$ satisfying (10) for all j . The case where $\bar{u}_j = u_{j-1}$ is trivial. For $u_j \neq \bar{u}_{j-1}$, by a suitable normalization we can take the data to be $\bar{u}_{j+1} \geq 1$, $\bar{u}_j = 1$, $\bar{u}_{j-1} = 0$ and $\bar{u}_{j-2} \leq 0$. A direct calculation of (3) gives

$$\bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1} = 1 - \lambda - \lambda\bar{u}_{j-2} + \lambda(1 - \lambda)(2s_{j-1} - s_j - s_{j-2})/2 \quad (21)$$

From (10), $|s_j| \leq 2$, $|s_{j-1}| \leq \text{Min}(-2\bar{u}_{j-2}, 2)$, $|s_{j-2}| \leq -2\bar{u}_{j-2}$. Under these bounds, it can be verified that the right hand side of (21) is positive for $0 \leq \lambda \leq 1$ and $\bar{u}_{j-2} \leq 0$. Thus monotonicity is preserved.

Note however that without condition (11), \bar{u}_j^{n+1} can lie outside of $[\bar{u}_{j-1}, u_j]$. So here we have an instance of the scheme being monotonicity preserving without being bounded by the initial data.

Reconstruction in Two Dimensions

For the two-dimensional advection equation

$$u_t + au_x + bu_y = 0 \quad (22)$$

with initial condition $u(x, y, 0) = u_0(x, y)$, a set of conditions under which higher order reconstructions preserve monotonicity is not known. The focus then shifts to schemes that are *positivity-preserving* (PP) ⁹⁻¹⁴ where some theoretical results can be proved. These are schemes that give positive updates when the initial data are positive, or equivalently, where the cell averages at time $n + 1$ are bounded by the cell averages at time level n . In the meteorology literature, such schemes are also referred to as positive definite.

In one dimension, the boundedness result used to prove Theorem-1, i.e. that

$$u_{j-1} \leq \bar{u}_j^{n+1} \leq \bar{u}_j \quad (23)$$

suffices to prove monotonicity. In two dimensions, this is no longer the case and PP schemes will not preserve monotonicity in general.

To introduce the reconstruction problem here, let us begin with a base scheme that uses midpoint rule in time and linear reconstructions in each cell. This scheme is chosen since: a) it can be readily extended to the Euler equations, b) it requires only one Riemann solver calculation per face per timestep and c) it has already been implemented in commercial codes¹⁵.

We assume a uniform grid of spacing Δx and Δy and $a \geq 0$ and $b \geq 0$. A linear reconstruction is assumed in each cell that can be written in normalized coordinates as

$$R_{i,j} = \bar{u}_{i,j} + S_{i,j}^x \xi + S_{i,j}^y \eta \quad (24)$$

with $|\xi|, |\eta| \leq 1/2$. The scheme can be written as

$$\bar{u}_{i,j}^{n+1} = \bar{u}_{i,j}^n - \lambda_x (u_{i+1/2,j}^{n+1/2} - u_{i-1/2,j}^{n+1/2}) - \lambda_y (u_{i,j+1/2}^{n+1/2} - u_{i,j-1/2}^{n+1/2}) \quad (25)$$

where $\lambda_x = a\Delta t/\Delta x$, $\lambda_y = b\Delta t/\Delta x$ and the interface values are obtained by taking a half timestep inside the cell and can be derived as

$$\begin{aligned} u_{i+1/2,j}^{n+1/2} &= \bar{u}_{i,j}^n + (1 - \lambda_x) S_{i,j}^x/2 - \lambda_y S_{i,j}^y/2 \\ u_{i,j+1/2}^{n+1/2} &= \bar{u}_{i,j}^n + (1 - \lambda_y) S_{i,j}^y/2 - \lambda_x S_{i,j}^x/2 \end{aligned} \quad (26)$$

The linear scheme with $S_{i,j}^x$ and $S_{i,j}^y$ given by their central difference values such as

$$S_{i,j}^x = (\bar{u}_{i+1,j} - \bar{u}_{i-1,j})/2 \quad (27)$$

is second order accurate in time and space and stable in the region $\lambda_x \geq 0$, $\lambda_y \geq 0$, $\lambda_x + \lambda_y \leq 1$. Thus the time step can be defined in terms of the CFL number by

$$\Delta t = CFL/(a/\Delta x + b/\Delta y) \quad (28)$$

We can now state the reconstruction problem in two dimensions:

What are sufficient conditions on the reconstruction $R_{i,j}(\xi, \eta)$ that will ensure that the scheme (25) is positivity preserving ?

For example, if we use any of the TVD limiters to calculate the x and y slopes, is the resulting scheme positivity preserving ? The answer to this depends on the limiter as the following example shows. Consider negative data given by

$$\begin{aligned} \bar{u}_{-1,1} &= -1, & \bar{u}_{0,1} &= 0 \\ \bar{u}_{-2,0} &= -100 & \bar{u}_{-1,0} &= -1, & \bar{u}_{0,0} &= 0 & \bar{u}_{1,0} &= 0 \\ \bar{u}_{-1,-1} &= -1 & \bar{u}_{0,-1} &= -10, & \bar{u}_{1,-1} &= -100 \\ \bar{u}_{0,-2} &= -100 \end{aligned} \quad (29)$$

For $\lambda_x = 0.6$, $\lambda_y = 0.2$, both the Superbee (46) and the Average limiter (45) (these limiters are defined in the appendix below) give $\bar{u}_{0,0}^{n+1} = 16/50$, which shows that these schemes do not preserve positivity. The Minmod limiter, however, gives $\bar{u}_{0,0}^{n+1} = -57/50$, which is acceptable. A closer look at the reconstructions in this example reveals that for the Superbee and Average limiters, the reconstructions in each cell fall outside the range of the

cell averages of its neighbors. This condition turns out to be sufficient to prove positivity as shown below.

Let us define $N_{i,j}$ to be the set of cell averages of the immediate first order neighbors of the cell (i, j) and $\bar{u}_{i,j}$.

$$N_{i,j} = \begin{bmatrix} \bar{u}_{i-1,j+1}, & \bar{u}_{i,j+1}, & \bar{u}_{i+1,j+1}, \\ \bar{u}_{i-1,j}, & \bar{u}_{i,j}, & \bar{u}_{i+1,j}, \\ \bar{u}_{i-1,j-1}, & \bar{u}_{i,j-1}, & \bar{u}_{i+1,j-1} \end{bmatrix} \quad (30)$$

and $U_{i,j}$ to be the range of variation of u on $N_{i,j}$.

$$U_{i,j} = [\text{Min}[N_{i,j}], \text{Max}[N_{i,j}]] \quad (31)$$

Theorem-3: Assume that for the scheme (25) the reconstruction in each cell is bounded by its immediate neighbors, i.e. for all (i, j)

$$R_{i,j} \in U_{i,j} \quad (32)$$

Then, for $\lambda_x + \lambda_y \leq 1$,

$$\bar{u}_{i,j}^{n+1} \in U_{i,j} \cup U_{i-1,j} \cup U_{i,j-1} \quad (33)$$

i.e. the cell average at the next timestep lies inside the union of the averages in the neighborhoods of (i, j) , $(i-1, j)$ and $(i, j-1)$.

Proof: Using (25) and (26), we can write the scheme as

$$\begin{aligned} \bar{u}_{i,j}^{n+1} &= (1 - \lambda_x - \lambda_y)(\bar{u}_{i,j} - \lambda_x S_{i,j}^x/2 - \lambda_y S_{i,j}^y/2) \\ &\quad + \lambda_x(\bar{u}_{i-1,j} + (1 - \lambda_x) S_{i-1,j}^x/2 - \lambda_y S_{i-1,j}^y/2) \\ &\quad + \lambda_y(\bar{u}_{i,j-1} - \lambda_x S_{i,j-1}^x/2 + (1 - \lambda_y) S_{i,j-1}^y/2) \end{aligned} \quad (34)$$

which can be written in terms of the normalized reconstructions as

$$\begin{aligned} \bar{u}_{i,j}^{n+1} &= (1 - \lambda_x - \lambda_y) R_{i,j}(-\lambda_x/2, -\lambda_y/2) \\ &\quad + \lambda_x R_{i-1,j}((1 - \lambda_x)/2, -\lambda_y/2) \\ &\quad + \lambda_y R_{i,j-1}(-\lambda_x/2, (1 - \lambda_y)/2) \end{aligned} \quad (35)$$

Under the stability limit $\lambda_x + \lambda_y \leq 1$, the points where the reconstructions are sampled in the above equation lie inside their respective cells. Thus the right hand side is a convex combination of sampled reconstruction values which are bounded by (32) and thus (33) is immediate. \square

It is easily verified that Theorem-3 holds true for all propagation directions provided the interface values (26) are defined from upwind cells. The general CFL limit is then $|\lambda_x| + |\lambda_y| \leq 1$.

The condition that the reconstruction be bounded over the whole cell by all its immediate neighbors (including corner neighbors) was introduced by Barth and Jespersen¹⁶ in the context of unstructured grids. The above result shows that this condition is useful for constructing PP schemes in multi-dimensions for structured meshes as well.

We remark that the choice of stencil, i.e. (30) used above is somewhat arbitrary. In fact, for the Minmod scheme, the reconstruction is so tightly constrained that $R_{i,j}$ actually lies inside the range of $\bar{u}_{i,j}$, and two linearly independent cells from $\bar{u}_{i\pm 1,j}$, and $\bar{u}_{i,j\pm 1}$. From this it follows that $\bar{u}_{i,j}^{n+1}$ is bounded by the four values $\bar{u}_{i,j}$, $\bar{u}_{i-1,j}$, $\bar{u}_{i,j-1}$, and $\bar{u}_{i-1,j-1}$, a much smaller stencil than (33).

In the next section, we derive an efficient limiter that enforces (32) but is not as dissipative as the Minmod limiter.

A PP Limiter

The procedure for modifying the slopes to satisfy (32) is far from unique. The approach described here is similar to the approach of Barth and Jespersen¹⁶, adapted to structured grids and modified so as to depend continuously on the data.

To satisfy (32) within a cell, the idea is to restrict the slopes so that the reconstructed values at the four corners of the cell lie inside the required interval, i.e., (31). Due to the symmetry of the grid, this is equivalent to restricting $|S_{i,j}^x| + |S_{i,j}^y|$ to lie inside another interval. We skip the details and give the result in algorithm form.

Let the slopes be defined initially by their centered difference values, namely

$$\begin{aligned} S_{i,j}^x &= (\bar{u}_{i+1,j} - \bar{u}_{i-1,j})/2 \\ S_{i,j}^y &= (\bar{u}_{i,j+1} - \bar{u}_{i,j-1})/2 \end{aligned} \quad (36)$$

and let V_{min} and V_{max} be defined by

$$V_{min} = \text{Min} \begin{bmatrix} \bar{u}_{i-1,j+1} - \bar{u}_{i,j}, & \bar{u}_{i,j+1} - \bar{u}_{i,j}, & \bar{u}_{i+1,j+1} - \bar{u}_{i,j}, \\ \bar{u}_{i-1,j} - \bar{u}_{i,j}, & -\epsilon, & \bar{u}_{i+1,j} - \bar{u}_{i,j}, \\ \bar{u}_{i-1,j-1} - \bar{u}_{i,j}, & \bar{u}_{i,j-1} - \bar{u}_{i,j}, & \bar{u}_{i+1,j-1} - \bar{u}_{i,j} \end{bmatrix} \quad (37)$$

$$V_{max} = \text{Max} \begin{bmatrix} \bar{u}_{i-1,j+1} - \bar{u}_{i,j}, & \bar{u}_{i,j+1} - \bar{u}_{i,j}, & \bar{u}_{i+1,j+1} - \bar{u}_{i,j}, \\ \bar{u}_{i-1,j} - \bar{u}_{i,j}, & \epsilon, & \bar{u}_{i+1,j} - \bar{u}_{i,j}, \\ \bar{u}_{i-1,j-1} - \bar{u}_{i,j}, & \bar{u}_{i,j-1} - \bar{u}_{i,j}, & \bar{u}_{i+1,j-1} - \bar{u}_{i,j} \end{bmatrix} \quad (38)$$

where ϵ is a small positive number ($\epsilon = 10^{-10}$ in all our numerical experiments). Then we define

$$V = 2 \frac{\text{Min}(|V_{min}|, |V_{max}|)}{(|S_{i,j}^x| + |S_{i,j}^y|)} \quad (39)$$

The final slopes are then

$$\begin{aligned} S_{i,j}^x &= \text{Min}(1, V) S_{i,j}^x \\ S_{i,j}^y &= \text{Min}(1, V) S_{i,j}^y \end{aligned} \quad (40)$$

It can be verified that after limiting, the reconstructed values in the cell (i, j) lie inside the interval $U_{i,j}$ and that the reconstruction depends continuously on the data.

The main problem with this and other limiters is the loss of accuracy at extrema which make these schemes only first order accurate in the Max. norm. If a scheme is required to be second order accurate everywhere (i.e. in the Max. norm), such a scheme will advect second order polynomials exactly. It follows that while advecting smooth extrema such a scheme will give updates not bounded by the data. It thus appears that second order accuracy in the Max. norm and positivity preservation are mutually exclusive.

This is not to say that some compromise might be found where positivity preservation may be relaxed in some useful way near extrema. However, such a compromise has eluded us thus far.

Numerical Experiments

We present some numerical experiments on two dimensional advection to assess the performance of PP schemes. We solve (22) on the domain $[-1,1] \times [-1,1]$. Periodic boundary conditions are imposed at the boundaries. A number of initial conditions, advection directions and timesteps were explored but only two cases are reported here. After computation, we scan the computed solution for overshoots and undershoots and present line plots through these regions. These represent the worst case results.

Example 1: In this example, a uniform grid of 120 X 120 cells is used with convection velocities $a = 0.8$ and $b = 0.1$. The timestep was calculated by (28) with $CFL = 0.8$ and the final time is $t = 20$, which corresponds to 8 periods in x and one period in y . The initial condition is

$$\begin{aligned} u(x, y, 0) &= 1 \quad \text{for } (x^2 + y^2)^{1/2} < 0.4 \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{41}$$

In Figure 2, the results along the line $y = -0.264$ ($j=45$) are shown. The solid line is the exact solution. As can be seen both the Superbee and the Average limiter give significant overshoots while the Minmod and PP limiters have no overshoots or undershoots. The PP limiter is not as dissipative as the Minmod limiter but shock resolution is still quite poor. In terms of efficiency, the PP limiter is in the same ball park as any of the TVD schemes.

Example-2: The initial condition here is

$$u(x, y, 0) = \exp(-\beta(x^2 + y^2)) \tag{42}$$

with $\beta = 200$. The convection velocities are $a = 1$ and $b = 1$. The same grid and CFL number are used as in Example-1, but the final time is $t = 2$ which corresponds to one period in both x and y directions. The results along the line $y = 0$ are shown in Figure 3.

where the loss of accuracy at extrema is clearly visible. The new PP limiter gives results similar to the Average limiter. The results from the Minmod limiter are much poorer and are not shown.

We remark that a large number of numerical experiments with various initial conditions seems to suggest that all the TVD limiters are positivity preserving when the advection direction is 45 degrees.

Conclusions

In this paper, we have considered anew what conditions a reconstruction must satisfy for an upwind scheme to have desirable nonoscillatory properties. In one dimension, a simple concise proof is presented that if any general reconstruction is increasing for increasing data and bounded by neighboring cell averages, the resulting scheme is monotonicity-preserving. In the special case of linear reconstruction it is shown that merely bounding the reconstruction between neighboring cell averages is sufficient for monotonicity preservation.

In two dimensions it is shown that if the reconstruction over the whole cell is bounded by the averages of all its first order neighbors, then the scheme is positivity preserving. An efficient limiter that achieves this is also presented.

We hope that these concise proofs will be useful in the teaching of upwind methods and also lead to the design of new schemes.

Appendix

For completeness, we summarize several popular one dimensional TVD limiters here. The reconstruction in each cell is assumed to be $R_j(\xi) = \bar{u}_j + s_j \xi$ and these limiters are various recipes to determine s_j from the data. Let s_+ and s_- denote the forward and backward differences given by

$$\begin{aligned} s_+ &= (\bar{u}_{j+1} - \bar{u}_j) \\ s_- &= (\bar{u}_j - \bar{u}_{j-1}) \end{aligned} \tag{43}$$

The Minmod limiter is given by

$$s_j = \frac{1}{2} [\text{sgn}(s_+) + \text{sgn}(s_-)] \text{Min}(|s_-|, |s_+|) \tag{44}$$

The Average limiter of Van Leer can be written as

$$s_j = \frac{1}{2} [\text{sgn}(s_+) + \text{sgn}(s_-)] \text{Min}((s_+ + s_-)/2, 2|s_-|, 2|s_+|) \tag{45}$$

and the Superbee limiter of Roe can be written as

$$s_j = \frac{1}{2} [\text{sgn}(s_+) + \text{sgn}(s_-)] \text{Min}(\text{Max}(|s_+|, |s_-|), 2|s_-|, 2|s_+|) \tag{46}$$

As mentioned above, the Minmod limiter is the most diffusive and the Superbee the least diffusive.

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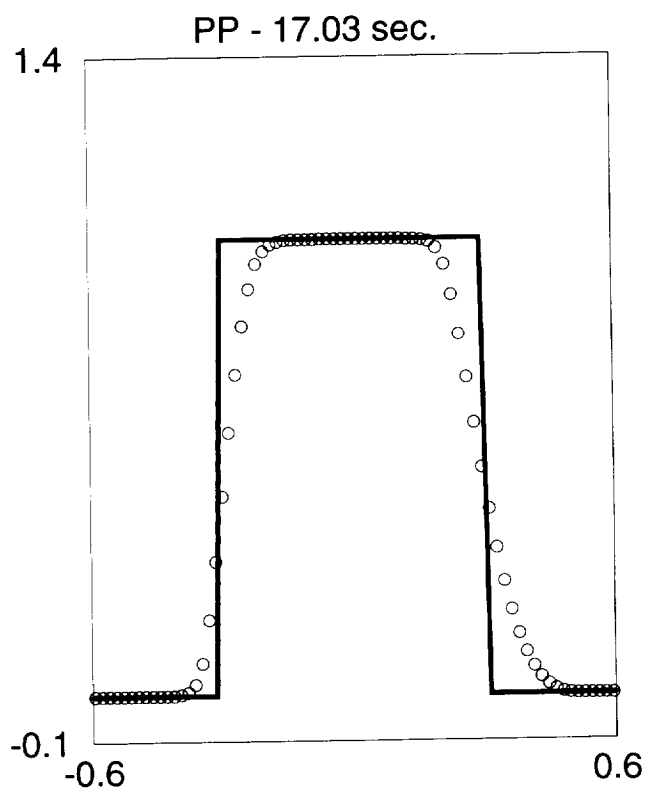
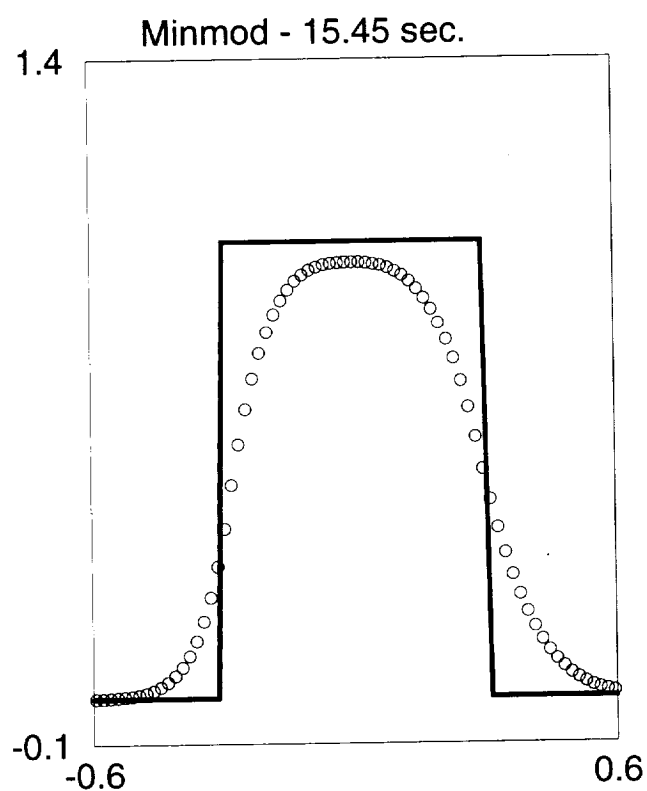
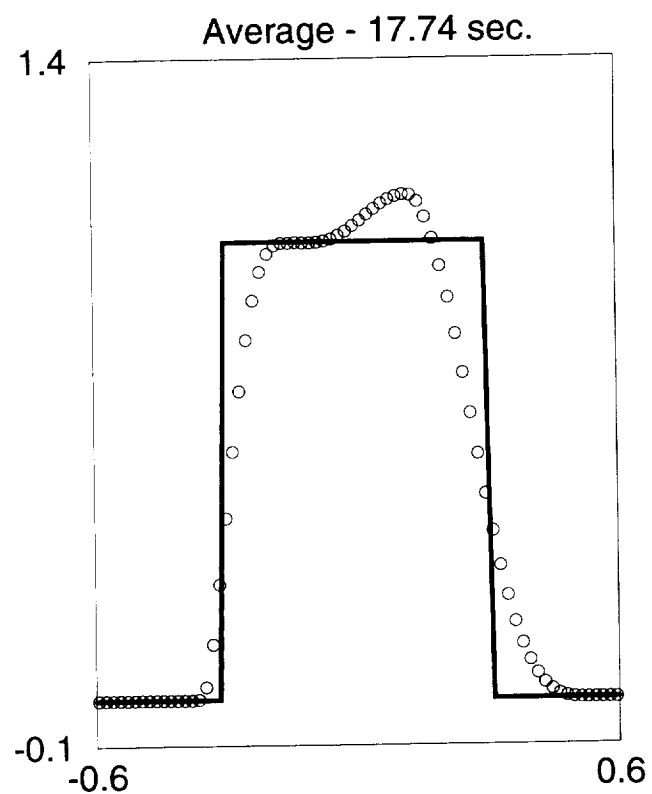
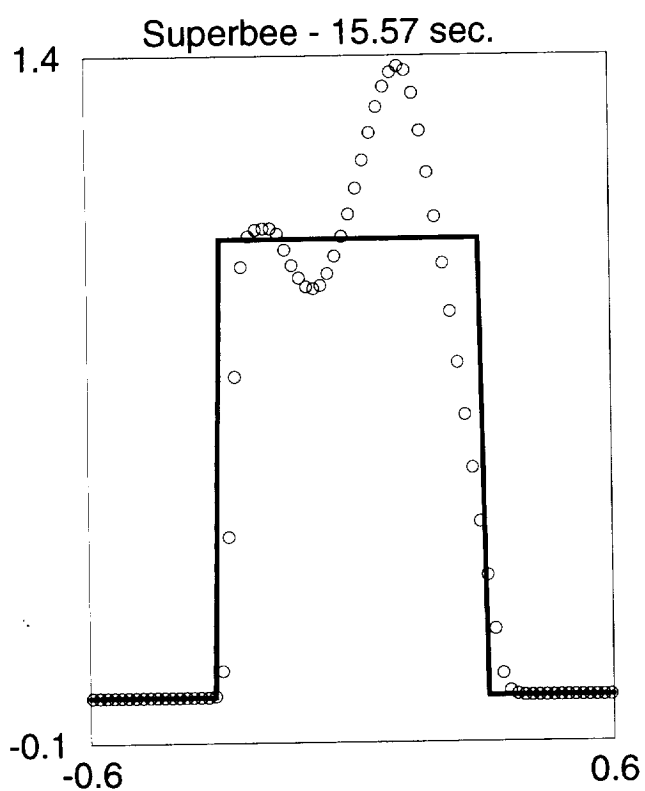


Figure 2: Advection of (41) with $a=0.8$, $b = 0.1$, $CFL = 0.8$, and $t = 20$. 120 X 120 uniform grid

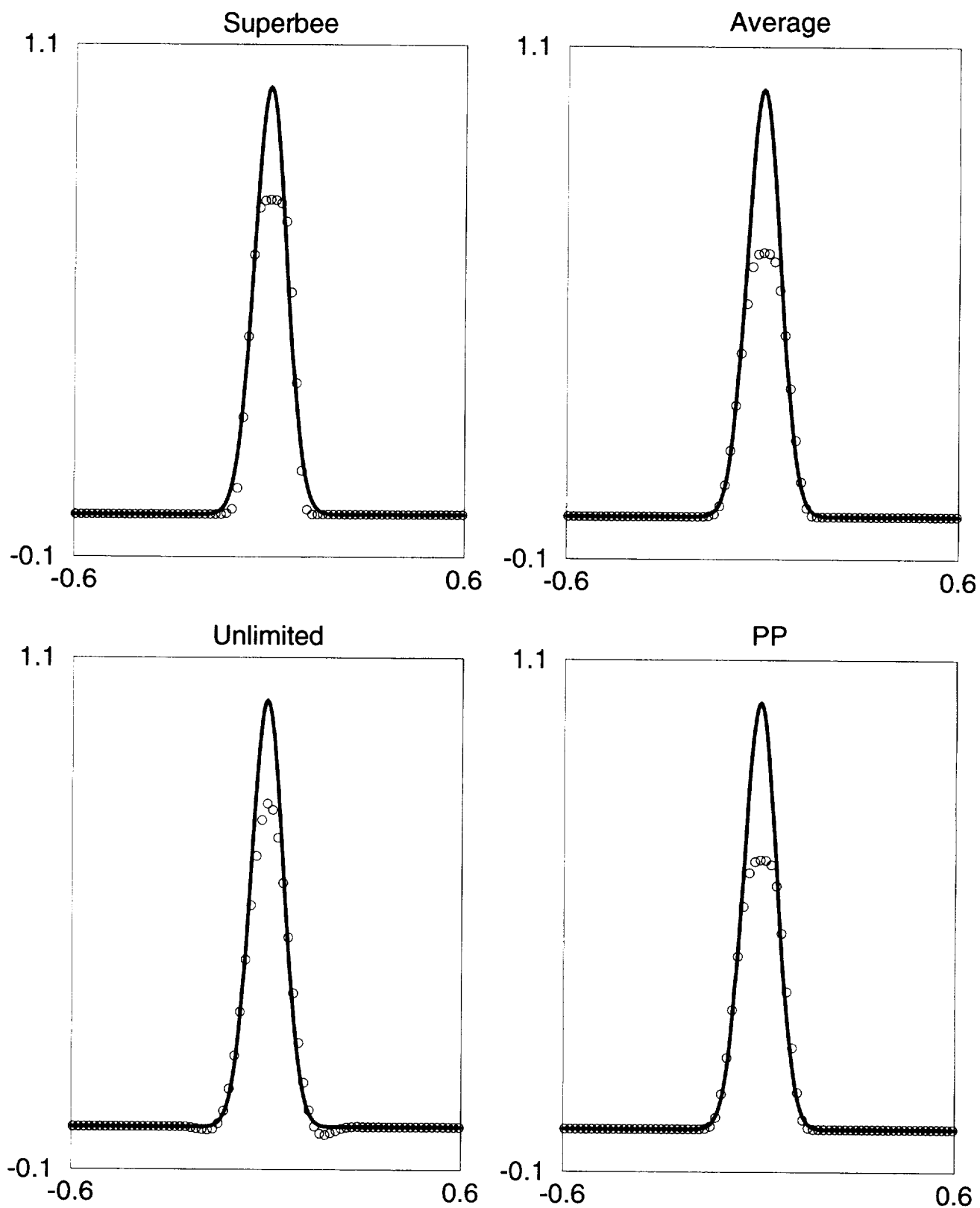


Figure 3: Advection of (42) with $a=1$, $b=1$, $CFL=0.8$, and $t=2$. 120 X 120 uniform grid

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13. ABSTRACT (Maximum 200 words) The role of reconstruction in avoiding oscillations in upwind schemes is reexamined, with the aim of providing simple, concise proofs. In one dimension, it is shown that if the reconstruction is any arbitrary function bounded by neighboring cell averages and increasing within a cell for increasing data, the resulting scheme is monotonicity preserving, even though the reconstructed function may have overshoots and undershoots at the cell edges and is in general not a monotone function. In the special case of linear reconstruction, it is shown that merely bounding the reconstruction between neighboring cell averages is sufficient to obtain a monotonicity preserving scheme. In two dimensions, it is shown that some 1D TVD limiters applied in each direction result in schemes that are not positivity preserving, i.e. do not give positive updates when the data are positive. A simple proof is given that if the reconstruction inside the cell is bounded by the neighboring cell averages (including corner neighbors), then the scheme is positivity preserving. A new limiter that enforces this condition but is not as dissipative as the Minmod limiter is also presented.	
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